University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

# Problem Set 7

**1.** Starting from the elements of $F^{\mu\nu}$ as given in PS 6 Problem 7, apply the metric tensor (twice) to find the elements of $F_{\mu\nu}$.

**2.** Define the contravariant field-strength tensor $F$ and the contravariant *dual* field-strength tensor $\mathcal{F}$ by

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$
$$\mathcal{F}^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \ ,$$

where $\epsilon^{\mu\nu\rho\sigma} = 1$ for $\mu\nu\rho\sigma = 0123$ or any permutation of 0123 that is achieved by an *even* number of interchanges of adjacent indices; $\epsilon^{\mu\nu\rho\sigma} = -1$ for $\mu\nu\rho\sigma = 1023$ or any other permutation of 0123 that is achieved by an *odd* number of interchanges of adjacent indices; and $\epsilon^{\mu\nu\rho\sigma} = 0$ otherwise. By explicit calculation, show that the elements of $\mathcal{F}^{\mu\nu}$ can be obtained from those of $F^{\mu\nu}$ by changing $\mathbf{E}$ into $c\mathbf{B}$ and $c\mathbf{B}$ into $-\mathbf{E}$.

**3.** By explicit evaluation, show that $\mathcal{F}^{\mu\nu}F_{\mu\nu}$ is proportional to $\mathbf{E}\cdot\mathbf{B}$, and find the constant of proportionality. (Because $\mathcal{F}^{\mu\nu}F_{\mu\nu}$ is obviously a Lorentz scalar, the Lorentz invariance of $\mathbf{E}\cdot\mathbf{B}$ is therefore said to be *manifest*.)

**4.** The two source-free Maxwell equations are equivalent to the single manifestly Lorentz-invariant equation

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0 \ .$$

Without making any reference to Maxwell's equations, using only formal manipulation, show that $\mathcal{F}$ has this property (*i.e.* its four-divergence vanishes). [*Hint*: Write $\mathcal{F}^{\mu\nu}$ in terms of $\epsilon^{\mu\nu\rho\sigma}$ and $F_{\rho\sigma}$. Then write $F_{\rho\sigma}$ in terms of $\partial_{\rho,\sigma}$ and $A_{\rho,\sigma}$. Finally, make use (twice) of the behavior of $\epsilon^{\mu\nu\rho\sigma}$ under interchange of adjacent indices.]

**5.** Is it possible to have an electromagnetic field that appears as a purely electric field in one inertial frame and a purely magnetic field in the other? What criteria must (uniform nonzero) $\mathbf{E}$ and $\mathbf{B}$ satisfy such that there exists an inertial frame in which the electromagnetic field is purely magnetic?

**6.** An infinitely long straight wire of negligible cross-sectional area moves in the $\hat{x}$ direction (parallel to its length) with speed $\beta c$ relative to the lab. As observed in its *rest* frame, the wire carries a uniform linear charge density $\lambda$ Coulombs/meter; in that frame, those charges are at rest. In the lab, write the elements of the field strength tensor at the point $(0, y, 0)$.

**7.** Consider a relativistic particle of mass $m$ and charge $e$ that accelerates in a uniform, static electric field with magnitude $E$ (there is no magnetic field). At $t = 0$ the particle is at rest. Solve for $\eta(t > 0)$, where $\eta \equiv \tanh^{-1}(\beta)$ is the particle's boost parameter or "rapidity".

**8.** Consider a relativistic particle of mass $m$ and charge $e$ that is in helical motion under the influence of a constant magnetic field of magnitude $B$ (there is no electric field). Its momentum component in the direction of the magnetic field is $p_0$. Show that the cyclotron angular frequency of this particle is

$$\Omega = \frac{eB}{\gamma_\perp m_{\text{eff}}} \ ,$$

where

$$\gamma_\perp \equiv \frac{1}{\sqrt{1-\beta_\perp^2}}$$
$$m_{\text{eff}} \equiv \sqrt{m^2 + p_0^2/c^2} \ ,$$

and $c\beta_\perp$ is the component of the particle's velocity that is perpendicular to the magnetic field. (That is, the transverse motion of a particle that moves in a helix is the same as that of a heavier particle that moves purely in a circle.)